CONCERNING A COMPOUND DISCONTINUOUS SOLUTION IN THE PROBLEM OF THE SURFACE OF REVOLUTION OF MINIMUM AREA

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Introduction. Among the minimum surfaces which can be represented by liquid films is a surface of revolution across which is a transverse plane circular film.* It can be constructed by placing one on the other two equal rings, dipping them into soap solution, and then drawing them apart always parallel to each other and perpendicular to an axis through their centers. A longitudinal section of the surface shows on each side of this axis a system of three curves meeting in a point, two of the curves extending respectively to the two rings, and the third extending to the axis. The experiment suggests the following mathematical problem:

Given two points A_0 and A_1 in the xy-plane and on the same side of the x-axis. Join the points by a curve confined to the positive side of the x-axis, and from an arbitrary point P_2 of this curve draw a curve to the x-axis. To find among all such systems of curves that one which when revolved about the x-axis shall determine a surface of revolution of minimum area. All curves which we consider will be of class D'.†

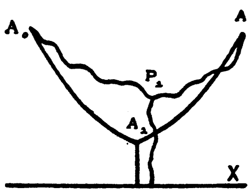

FIG. 1.

The object of the present paper is to give a solution of this problem. The discussion is divided into three parts A, B, C, dealing, respectively, with the necessary conditions for the solution, the sufficient conditions, and the character of the field of extremals. The principal results obtained are as follows:

1. The solution must consist of two catenaries $A_0\,A_2$, $A_2\,A_1$, with the x-axis for directrix, and the normal $A_2\,M$ from A_2 to the x-axis. The catenaries must meet the normal $A_2\,M$ under an angle of 120°. (A, §§ 1–3.)

2. On the arc $A_0\,A_2$ there exists a critical point A_0^*, a limit on the left

* See Plateau, *Statiques des liquides*, for a general discussion of minimum surfaces which can be represented by liquid films.

† Bolza, *Lectures on the Calculus of Variations*, §2. A curve is of class D' if it is continuous and consists of a finite number of arcs each of which has a continuously turning tangent.

for the position of A_0; on the arc $A_2\,A_1$ there exists a conjugate point A_0', a limit on the right for the position of A_1. For the latter point there exists a simple geometric construction which we shall call the Lindelöf construction for our problem. (A, §§ 4–9.)

3. The above are not only necessary but also sufficient conditions for a relative minimum (B).

4. The solution is unique within a field defined by the one-parameter set of extremal-systems passing through A_0, defined in 1 above. (C, §§ 1–3.)

5. The field consists of two parts in which the area of the surface of revolution given by our solution is respectively greater or smaller than that given by the solution found by Goldschmidt.* (C, §4.)

6. Experiment verifies the results defining the position of the conjugate point.† (C, §5.)

A. NECESSARY CONDITIONS FOR THE SOLUTION.

1. The system consisting of segments of two catenaries and one straight line. We suppose a totality $\mathfrak{M}$ of systems of three curves of the form $x = x(t)$, $y = y(t)$, which, as indicated above,

1) are of class D',

2) lie in the region $y > 0$,

3) meet in a point P_2, joining it respectively to the given points A_0 and A_1, and the x-axis.

We suppose further that a particular system through a point A_2, of which we shall denote the segment $A_0\,A_2$ by $\mathfrak{E}_0$, the segment $A_2\,A_1$ by $\mathfrak{E}_1$, and the segment joining A_2 to the x-axis by $\mathfrak{E}_2$, actually furnishes a relative minimum for the integral

$$I = \int_{\mathfrak{E}_0} 2\pi y\sqrt{x'^2 + y'^2}\,dt + \int_{\mathfrak{E}_1} 2\pi y\sqrt{x'^2 + y'^2}\,dt + \int_{\mathfrak{E}_2} 2\pi y\sqrt{x'^2 + y'^2}\,dt,$$

where x' and y' are the derivatives dx/dt and dy/dt.

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* Goldschmidt, *Prize Essay*, 1830.

† After the completion of this paper, two papers by H. Tallqvist dealing with the same problem came to my attention: "Determination experimentale de la limite de stabilité de quelques snrfaces minima." *Transactions of the Scientific Society of Finland*, vol. 32 (1889); and *Bestimmung einiger Minimalflächen deren Begrenzung gegeben ist*, Helsingfors, 1890. Tallqvist's results concerning the limit of stability, obtained by an entirely different method, agree with our own results in the special case of equal rings; they are wrong in the case of unequal rings, as will be shown in C, §5, which I have added for the sake of comparison.

We vary each of the segments $\mathfrak{C}_0$, $\mathfrak{C}_1$, $\mathfrak{C}_2$ separately, leaving the remainder of the system fixed. By the usual theory, each of these curves must in the first place be a solution of the Euler equation

$$F_{xy'} - F_{x'y} + F_1\,(x'y'' - x''y') = 0,$$

where
$$F = 2\pi y\,\sqrt{x'^2 + y'^2},$$

and therefore for our problem must be either a catenary,*

$$x = at + \beta, \qquad y = a\,\mathrm{ch}\,t,$$

or a vertical straight line,

$$x = a, \qquad y = t.$$

In particular $\mathfrak{C}_0$ and $\mathfrak{C}_1$ must be catenaries, and $\mathfrak{C}_2$ must be a straight line, since no catenary of the set given above touches the x-axis. The points A_i have coordinates $(a_i,\, b_i)$, and we write

$$(1)\qquad \begin{array}{llll}
\mathfrak{C}_0: & \begin{array}{l} x = \alpha_0 t + \beta_0, \\ y = \alpha_0 \mathrm{ch}\, t, \end{array} \text{ where } \begin{cases} a_0 = \alpha_0 t_0 + \beta_0, \\ b_0 = \alpha_0 \mathrm{ch}\, t_0, \end{cases} & \text{and} & \begin{cases} a_2 = \alpha_0 t_2 + \beta_0, \\ b_2 = \alpha_0 \mathrm{ch}\, t_2; \end{cases} \\[2ex]
\mathfrak{C}_1: & \begin{array}{l} x = \alpha_1 \tau + \beta_1, \\ y = \alpha_1 \mathrm{ch}\tau, \end{array} \text{ where } \begin{cases} a_2 = \alpha_1 \tau_2 + \beta_1, \\ b_2 = \alpha_1 \mathrm{ch}\, \tau_2, \end{cases} & \text{and} & \begin{cases} a_1 = \alpha_1 \tau_1 + \beta_1, \\ b_1 = \alpha_1 \mathrm{ch}\tau_1; \end{cases} \\[2ex]
\mathfrak{C}_2: & x = a_2. & &
\end{array}$$

Further, $\mathfrak{C}_0$, $\mathfrak{C}_1$, $\mathfrak{C}_2$ must each satisfy the conditions numbered by Bolza I, II, III, and IV,† which are necessary for a minimum. The stronger forms II′ and IV′ are always satisfied‡ since along the system of curves under consideration

$$F_1(xy \cos\gamma \sin\gamma) = \frac{F_{x'x'}}{y'^2} = y > 0.$$

We may also immediately require that condition III be satisfied in the stronger form III′, by the arcs $\mathfrak{C}_0$ and $\mathfrak{C}_1$, since the envelope of the one-parameter set of catenaries through A_2 has no singular point, and therefore

* Bolza, loc. cit., p. 153.

† Loc. cit., §25–28.

‡ Bolza, loc. cit., p. 146.

the arc $\mathfrak{E}_0$ from A_2 to its conjugate A_2' could never furnish a minimum.* A similar remark holds for $\mathfrak{E}_1$. *The two catenaries and the normal to the x-axis, which form the system, must therefore separately satisfy all the conditions for fixed end points.*

2. The integral as a function of the coordinates of P_2, the corner point. We shall apply the following theorem:†

Let $M_0(t = t_0,\ x = a_0,\ y = b_0)$ and $M_1(t = t_1,\ x = a_1,\ y = b_1)$ be two non-conjugate points on any extremal. Then it is always possible to construct through the points $P_0(x = x_0,\ y = y_0)$ and $P_1(x = x_1,\ y = y_1)$ in the neighborhood of M_0 and M_1 respectively, a unique extremal, over which the integral is a single-valued, continuous function of the coordinates of P_0 and P_1,

$$J(x_0,\ y_0,\ x_1,\ y_1),$$

with continuous partial derivatives in the vicinity of $(a_0,\ b_0,\ a_1,\ b_1)$. Further, the total differential of J is ‡

$$dJ(x_0,\ y_0,\ x_1,\ y_1) = F_{x'}(x_1,\ y_1,\ x_1',\ y_1')\,dx_1 + F_{y'}(x_1,\ y_1,\ x_1',\ y_1')\,dy_1 - F_{x'}(x_0,\ y_0,\ x_0',\ y_0')\,dx_0 - F_{y'}(x_0,\ y_0,\ x_0',\ y_0')\,dy_0.$$

For our problem, therefore,

$$(2)\quad dJ(x_0,\ y_0,\ x_1,\ y_1) = 2\pi\left\{\frac{y_1x_1'}{\sqrt{x_1'^2 + y_1'^2}}\,dx_1 + \frac{y_1y_1'}{\sqrt{x_1'^2 + y_1'^2}}\,dy_1 - \frac{y_0x_0'}{\sqrt{x_0'^2 + y_0'^2}}\,dx_0 - \frac{y_0y_0'}{\sqrt{x_0'^2 + y_0'^2}}\,dy_0\right\}.$$

For the extremal-system, $\mathfrak{E}_0$, $\mathfrak{E}_1$, $\mathfrak{E}_2$, the integral I is completely determined in terms of this function J, its value being

$$(3)\qquad I(\mathfrak{E}_0,\ \mathfrak{E}_1,\ \mathfrak{E}_2) = J(a_0,\ b_0,\ a_2,\ b_2) + J(a_2,\ b_2,\ a_1,\ b_1) + \pi\, b_2^2.$$

Suppose $P_2(x_2,\ y_2)$ any point in the vicinity of A_2. Then if the catenary through A_0 and P_2 be $\overline{\mathfrak{E}}_0$, and that through P_2 and A_1 be $\overline{\mathfrak{E}}_1$, we have

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* Bolza, loc. cit., p. 204.

† Bolza, loc. cit., p. 174, footnote 1.

‡ Bolza, loc. cit., p. 176.

$$
(4) \qquad
\begin{aligned}
\overline{\mathfrak{E}}_0: &\quad x = \bar{a}_0 \bar{t} + \bar{\beta}_0, \\
&\quad y = \bar{a}_0 \operatorname{ch} \bar{t}; \\
\overline{\mathfrak{E}}_1: &\quad x = \bar{a}_1 \bar{\tau} + \bar{\beta}_1, \\
&\quad y = \bar{a}_1 \operatorname{ch} \bar{\tau}; \\
\overline{\mathfrak{E}}_2: &\quad x = x_2,
\end{aligned}
$$

with equations similar to those given under (1) at the end points. Over this system, the integral I is a function of x_2, y_2, namely:

$$
(5) \quad I(\overline{\mathfrak{E}}_0, \overline{\mathfrak{E}}_1, \overline{\mathfrak{E}}_2) = \phi(x_2, y_2) = J(a_0, b_0, x_2, y_2) + J(x_2, y_2, a_1, b_1) + \pi y_2^2.
$$

It is now necessary that the function $\phi(x_2, y_2)$ have a minimum for the values $x_2 = a_2$, $y_2 = b_2$. By the ordinary theory of maxima and minima, the following conditions must be satisfied for $x_2 = a_2$, $y_2 = b_2$:

$$
(6) \quad \frac{\partial \phi}{\partial x_2} = 0, \quad \frac{\partial \phi}{\partial y_2} = 0, \quad \frac{\partial^2 \phi}{\partial x_2 \partial x_2} \gtreqless 0, \quad \left(\frac{\partial^2 \phi}{\partial x_2 \partial y_2}\right)^2 - \left(\frac{\partial^2 \phi}{\partial x_2 \partial x_2}\right)\left(\frac{\partial^2 \phi}{\partial y_2 \partial y_2}\right) \leqq 0.
$$

3. The corner condition. We first discuss the conditions on the first derivatives of ϕ with respect to x_2 and y_2, where $x_2 = a_2$ and $y_2 = b_2$. From (2) and (5), we have

$$
\frac{\partial \phi}{\partial x_2} = 2\pi \frac{y_2 x_2'}{\sqrt{x_2'^2 + y_2'^2}} - 2\pi \frac{y_2 \tilde{x}_2'}{\sqrt{\tilde{x}_2'^2 + \tilde{y}_2'^2}},
$$

$$
\frac{\partial \phi}{\partial y_2} = 2\pi \frac{y_2 y_2'}{\sqrt{x_2'^2 + y_2'^2}} - 2\pi \frac{y_2 \tilde{y}_2'}{\sqrt{\tilde{x}_2'^2 + \tilde{y}_2'^2}} + 2\pi y_2,
$$

where x_2', y_2' refer to $\overline{\mathfrak{E}}_0$ and $\tilde{x}_2'$, $\tilde{y}_2'$ refer to $\overline{\mathfrak{E}}_1$. From (4), we see that for $x_2 = a_2$ and $y_2 = b_2$,

$$
x_2' = a_0, \quad y_2' = a_0 \operatorname{sh} t_2, \quad \sqrt{x_2'^2 + y_2'^2} = y_2,
$$

and

$$
\tilde{x}_2' = a_1, \quad \tilde{y}_2' = a_1 \operatorname{sh} \tau_2, \quad \sqrt{\tilde{x}_2'^2 + \tilde{y}_2'^2} = y_2.
$$

Hence at the point (a_2, b_2) we must have

$$
(7) \qquad \frac{\partial \phi}{\partial x_2} = 2\pi(a_0 - a_1) = 0,
$$

and hence

(8) $$a_0 = a_1.$$

Moreover, (6b) requires

(9) $$\frac{\partial\phi}{\partial y_2} = 2\pi a_0(\operatorname{sh} t_2 - \operatorname{sh} \tau_2 + \operatorname{ch} t_2) = 0.$$

But
$$b_2 = a_0 \operatorname{ch} t_2 = a_1 \operatorname{ch} \tau_2,$$
and therefore
$$t_2 = \pm\, \tau_2.$$
Using the upper sign, (9) becomes impossible, since $a_0 \neq 0$ and $\operatorname{ch} t_2 \neq 0$. Hence

(10) $$t_2 = -\, \tau_2,$$

and the equations
$$a_2 = a_0 t_2 + \beta_0 = a_1 \tau_2 + \beta_1$$
give

(11) $$a_2 = \tfrac{1}{2}(\beta_0 + \beta_1).$$

Condition (9) now requires
$$2 + \coth t_2 = 0;$$
from which, if we represent by p_2 the slope of $\mathfrak{C}_0$ at A_2 and by π_2 that of $\mathfrak{C}_1$ at A_2, we have

(12) $$t_2 = -\log\sqrt{3}, \qquad p_2 = \operatorname{sh} t_2 = -\frac{1}{\sqrt{3}},$$
$$\tau_2 = \log\sqrt{3}, \qquad \pi_2 = \operatorname{sh}\tau_2 = +\frac{1}{\sqrt{3}}.$$

We obtain from equations (8) and (12), therefore, the result that *the vertices of $\mathfrak{C}_0$ and $\mathfrak{C}_1$ must be at the same distance from the x-axis, and that the three curves $\mathfrak{C}_0$, $\mathfrak{C}_1$, $\mathfrak{C}_2$ must all meet at an angle of* 120°.*

4. Critical points. We shall now discuss the remaining conditions of (6).

* Physically, a result of equal tension; Tallqvist, *Determination experimentale* . . ., loc. cit., p. 7. Compare also Plateau, loc. cit., vol. 1, §173, p. 294, where the same result appears for a system of two equal spherical surfaces and their plane of intersection.

We obtain partial derivatives of $\bar{a}_0$, $\bar{a}_1$, $\bar{t}_2$, $\bar{\tau}_2$ with respect to x_2 and y_2 from equations for the end points of $\mathfrak{E}_0\,\mathfrak{E}_1$ similar to those given under (1). Evidently

$$x_2 - a_0 = \bar{a}_0(\bar{t}_2 - \bar{t}_0),$$
$$(13) \qquad b_0 = \bar{a}_0 \operatorname{ch} \bar{t}_0,$$
$$y_2 = \bar{a}_0 \operatorname{ch} \bar{t}_2.$$

Differentiating (13) with respect to x_2, and putting $x_2 = a_2$, $y_2 = b_2$, we obtain

$$1 = \frac{\partial a_0}{\partial x_2}(t_2 - t_0) + a_0\left(\frac{\partial t_2}{\partial x_2} - \frac{\partial t_0}{\partial x_2}\right),$$
$$0 = \frac{\partial a_0}{\partial x_2} \operatorname{ch} t_0 + a_0 \operatorname{sh} t_0 \frac{\partial t_0}{\partial x_2},$$
$$0 = \frac{\partial a_0}{\partial x_2} \operatorname{ch} t_2 + a_0 \operatorname{sh} t_2 \frac{\partial t_2}{\partial x_2};$$

whence,

$$\frac{\partial a_0}{\partial x_2} = \frac{1}{t_2 - t_0 - \coth t_2 + \coth t_0}.$$

We shall use the abbreviation $\chi(t) = \coth t - t$.
Then

$$(14) \qquad \frac{\partial a_0}{\partial x_2} = \frac{1}{\chi(t_0) - \chi(t_2)}.$$

Differentiating (13) with respect to y_2, and putting $x_2 = a_2$, $y_2 = b_2$, we obtain

$$0 = \frac{\partial a_0}{\partial y_2}(t_2 - t_0) + a_0\left(\frac{\partial t_2}{\partial y_2} - \frac{\partial t_0}{\partial y_2}\right),$$
$$0 = \frac{\partial a_0}{\partial y_2} \operatorname{ch} t_0 + a_0 \operatorname{sh} t_0 \frac{\partial t_0}{\partial y_2},$$
$$1 = \frac{\partial a_0}{\partial y_2} \operatorname{ch} t_2 + a_0 \operatorname{sh} t_2 \frac{\partial t_2}{\partial y_2};$$

whence

$$(15) \qquad \frac{\partial a_0}{\partial y_2} = -\frac{1}{\chi(t_0) - \chi(t_2)} \cdot \frac{1}{\operatorname{sh} t_2} = -\frac{1}{\operatorname{sh} t_2}\frac{\partial a_0}{\partial \chi_2}.$$

Similarly, we obtain

$$(16)\qquad \frac{\partial a_1}{\partial x_2} = \frac{1}{\chi(\tau_1) - \chi(\tau_2)}, \qquad \frac{\partial a_1}{\partial y_2} = -\frac{1}{\operatorname{sh}\tau_2}\cdot\frac{\partial a_1}{\partial x_2}.$$

Further,

$$(17)\qquad \frac{\partial t_2}{\partial y_2} = \frac{1}{a_0 \operatorname{sh} t_2}\left[1 + \frac{\operatorname{ch} t_2}{\operatorname{sh} t_2}\cdot\frac{\partial a_0}{\partial x_2}\right], \qquad \frac{\partial \tau_2}{\partial y_2} = \frac{1}{a_1 \operatorname{sh}\tau_2}\left[1 + \frac{\operatorname{ch}\tau_2}{\operatorname{sh}\tau_2}\cdot\frac{\partial a_1}{\partial x_2}\right].$$

Making use of (15), (16), (17), we now obtain the second derivatives at the point (a_2, b_2):

$$\frac{\partial^2\phi}{\partial x_2\partial x_2} = 2\pi\left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right),$$

$$(18)\qquad \frac{\partial^2\phi}{\partial x_2\partial y_2} = 2\pi\left(-\frac{1}{\operatorname{sh} t_2}\frac{\partial a_0}{\partial x_2} + \frac{1}{\operatorname{sh}\tau_2}\frac{\partial a_1}{\partial x_2}\right),$$

$$\frac{\partial^2\phi}{\partial y_2\partial y_2} = 2\pi\left(1 + \coth t_2 - \coth\tau_2 + \frac{1}{\operatorname{sh} t_2}\frac{\partial a_0}{\partial x_2} - \frac{1}{\operatorname{sh}\tau_2}\frac{\partial a_1}{\partial x_2}\right).$$

Condition (6c) therefore requires

$$(19)\qquad \left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right) = \frac{1}{\chi(t_0) - \chi(t_2)} + \frac{1}{\chi(\tau_2) - \chi(\tau_1)} \geqq 0.$$

But $\chi(t)$ is a decreasing function, continuous except at $t = 0$. Hence, since $t_0 < t_2 < 0$ and $0 < \tau_2 < \tau_1$, each fraction is positive, and *condition* (6c) *is always fulfilled in the stronger form*

$$\left.\frac{\partial^2 J}{\partial x_2\partial x_2}\right|^{x_2 = a_0} > 0.$$

Since $-\operatorname{sh} t_2 = \operatorname{sh}\tau_2 = 1/\sqrt{3}$, condition (6d) requires that for $x_2 = a_2$, $y_2 = b_2$,

$$3\left(\frac{\partial a_0}{\partial x_2} + \frac{\partial a_1}{\partial x_2}\right)^2 - \left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right)\left[-3 + 3\left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right)\right] \lesseqgtr 0,$$

or

$$4\,\frac{\partial a_0}{\partial x_2}\cdot\frac{\partial a_1}{\partial x_2}+\frac{\partial a_0}{\partial x_2}-\frac{\partial a_1}{\partial x_2}\gtreqless 0.$$

If we apply (15) and (16), this becomes

$$\frac{\log 3+\chi(\tau_1)-\chi(t_0)}{\{\chi(\tau_1)-\chi(\tau_2)\}\{\chi(t_0)-\chi(t_2)\}}\gtreqless 0.$$

Since the denominator is always negative, this condition is equivalent to the following:

(20) $$\log 3+\chi(\tau_1)-\chi(t_0)\geqq 0,$$

from which two results may be derived.

Let t_0 have any fixed value, and let τ_1 increase from τ_2 on. Then since $\chi(t)$ is a decreasing function, $\log 3+\chi(\tau_1)-\chi(t_0)$ decreases from $\log 3+\chi(\tau_2)-\chi(t_0)$. Hence, if the latter value is negative, it is impossible to satisfy (20). But

$$\chi(\tau_2)=2-\log\sqrt{3},$$

and hence

$$\log 3+\chi(\tau_2)-\chi(t_0)=2+\log\sqrt{3}-\chi(t_0).$$

It is therefore necessary that

(21) $$\chi(t_0)\leqq 2+\log\sqrt{3}.$$

If we denote by t_0^* the negative value of t satisfying the equation,

(22) $$\chi(t)=2+\log\sqrt{3},$$

it is then necessary that

(23) $$t_0\gtreqqless t_0^*.$$

The above condition (22) being satisfied, condition (20) further requires that τ_1 satisfy the condition

(24) $$\chi(\tau_1)\geqq-\log 3+\chi(t_0).$$

If we denote by t_0' the positive value of t satisfying the equation

(25) $$\chi(t)=-\log 3+\chi(t_0),$$

it is then necessary that

(26) $$\tau_1\lesseqqgtr t_0'.$$

For the extreme value $t_0 = t_0^*$, t_0' must be equal to τ_2. For the extreme value $t_0 = t_2$, t_0' takes the value t_1^* which is the positive value of τ satisfying the equation

$$\chi(\tau) = \chi(t_2) - \log 3 = -2 - \log \sqrt{3}.$$

As t_0 increases from t_0^* to t_2, t_0' increases from τ_2 to τ_1^*.

5. Geometric construction for critical point. For these conditions restricting the position of A_0 and A_1 we have the following simple geometric construction:

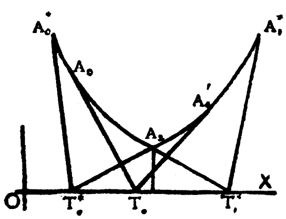

FIG. 2.

Let OT_0 be the x-intercept of the tangent to $\mathfrak{E}_0$ at A_0. Then

$$OT_0 = \beta_0 + a t_0 - a \coth t_0.$$

Let OT be the x-intercept of the tangent to $\mathfrak{E}_1$ at A_1. Then

$$OT = \beta_1 + a\tau - a \coth \tau.$$

Hence

$$\begin{aligned} OT_0 - OT &= \beta_0 - \beta_1 + a[t_0 - \tau - \coth t_0 + \coth \tau] \\ &= a(\tau_2 - t_2) + a[\chi(\tau) - \chi(t_0)] \\ &= a[\log 3 + \chi(\tau) - \chi(t_0)]. \end{aligned}$$

The point T coincides with T_0 if and only if

$$\log 3 + \chi(\tau) - \chi(t_0) = 0. \tag{27}$$

Hence the point A_0' $(\tau = t_0')$ is the point on $\mathfrak{E}_1$ determined by the tangent from T_0. We may say that A_0' is the conjugate to A_0 with respect to the broken-extremals $\mathfrak{E}_0\mathfrak{E}_1$, and may call our construction *the Lindelöf construction** for the discontinuous solution.

* Lindelöf, *Mathematische Annalen*, vol. 2 (1870), p. 160.

Let, further, the tangent to $\mathfrak{E}_1$ at A_2 cut the x-axis in T_0^*, and let A_0^* be the point of $\mathfrak{E}_0$ determined by the tangent from T_0^*. Similarly the tangent to $\mathfrak{E}_0$ at A_2 cuts the x-axis in a point T_1^*, and A_1^* is the point of $\mathfrak{E}_1$ determined by the tangent from T_1^*. As t_0 varies from t_0^* to t_2, A_0 describes the curve $\mathfrak{E}_0$ from A_0^* to A_2, and at the same time A_0' describes $\mathfrak{E}_1$ from A_2 to A_1^*.

6. The envelope of the broken-extremals. Let us consider the one-parameter family of broken-extremals $(\mathfrak{E}_0\mathfrak{E}_1)$,

$$(28) \qquad \begin{aligned} &\mathfrak{E}_0: \quad x = a_0 + a(t - t_0), \quad y = a \operatorname{ch} t, \quad t_0 \leqq t \leqq t_2, \\ &\mathfrak{E}_1 \quad\; x = x_2 + a(\tau + t_2), \quad y = a \operatorname{ch} \tau, \quad t_2 \leqq \tau \leqq t_0', \end{aligned}$$

with a as the variable parameter, each of which passes through the point A_0 and satisfies the condition

$$\operatorname{sh} t_2 = \frac{-1}{\sqrt{3}}$$

at the corner point P_2 (x_2, y_2), where $t = t_2$ and $\tau = - t_2$. The value t_0 is determined by the condition

$$b_0 = a \operatorname{ch} t_0,$$

and we suppose that

$$t_0^* < t_0 \leqq - \log \sqrt{3},$$

so that A_0 is never coincident with the point A_0^* on $\mathfrak{E}_0$.

From the above equations we obtain for the coordinates of P_2:

$$(29) \qquad K: \quad \begin{aligned} &x_2 = a_0 + a(- \log \sqrt{3} - t_0), \\ &y_2 = \frac{2}{\sqrt{3}} a, \qquad \text{where} \qquad b_0 = a \operatorname{ch} t_0. \end{aligned}$$

Since x_2 and y_2 are continuous functions of a, P_2 describes a continuous curve K, which we shall call *the corner curve*.

The set of catenaries $\mathfrak{E}_1$ possesses an envelope, E, given by the second equation of (28) together with the following condition on their functional determinant:

$$\frac{\partial(xy)}{\partial(a\tau)} = 0.$$

The latter gives

$$
(30) \qquad \frac{\partial(xy)}{\partial(a\tau)} = \begin{vmatrix} a & a\,\mathrm{sh}\,\tau \\ \tau - t_0 - \log 3 - \dfrac{y_0}{\sqrt{y_0^2 - \bar{a}^2}} & \mathrm{ch}\,\tau \end{vmatrix}
$$

$$
= a\,\mathrm{sh}\,\tau\left\{\coth\tau - \tau + \log 3 - t_0 + \frac{y_0}{\sqrt{y_0^2 - \bar{a}^2}}\right\}
$$

$$
= a\,\mathrm{sh}\,\tau\,\{\chi(\tau) - \chi(t_0) + \log 3\} = 0.
$$

It therefore follows for a particular system of the set $\mathfrak{E}_0\,\mathfrak{E}_1\,\mathfrak{E}_2$ that $A_0'(\tau = t_0')$ *is the point in which the particular catenary* $\mathfrak{E}_1$ *touches the envelope of the set* $\mathfrak{E}_1$. *We shall call this also the envelope of the set of broken-extremals* $(\mathfrak{E}_0\ \ \mathfrak{E}_1)$ *through* A_0.

7. The physical illustration of the critical point. A physical illustration is now of interest. If we construct a film between two rings as indicated in the introduction, the film is stable so long as the rings are sufficiently near together. At a fixed distance, however, the film becomes unstable, and divides into two separate bubbles which contract into plane circular films.* This point of instability is indeed the conjugate point.

We way readily obtain numerical values from our formulas.

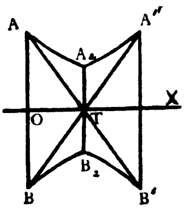

FIG. 3.

Consider the longitudinal section of the film, which for equal rings is symmetrical with respect to the point of intersection of the axis and the transverse film. Let its axis coincide with the x-axis, and the diameter AB of one ring with the y-axis. Let $A'B'$ be the diameter of the second ring and A_2B_2 that of the transverse film. Let $(0, 1)$ be the point A_0 and (x_2, y_2) the point A_2. Then $(2x_2, 1)$ is the point A'. The tangents to the arcs AA_2 and A_2A' at A and A' respectively meet at $T(x_2, 0)$. For the catenary AA_2,

$$
\mathfrak{E}_0: \qquad \begin{aligned} y &= a\,\mathrm{ch}\,t, \\ b_0 &= 1 = a\,\mathrm{ch}\,t_0, \\ x &= a(t - t_0). \end{aligned}
$$

* The Goldschmidt discontinuous solution.

The coordinates of A_2 are therefore given by the equations

$$A_2: \quad \begin{aligned} x_2 &= a(t_2 - t_0) = -\frac{b_0}{b_0'} = -a\,\frac{\operatorname{ch} t_0}{\operatorname{sh} t_0}, \\ y_2 &= a \operatorname{ch} t_2, \\ t_2 &= -\log\sqrt{3}, \qquad \text{where} \qquad 1 = a \operatorname{ch} t_0. \end{aligned}$$

Hence
$$\coth t_0 - t_0 = \log\sqrt{3} = .5493$$
and
$$t_0 = -\,1.6292.$$
It follows that
$$a = \frac{1}{\operatorname{ch} t_0} = .3777,$$
$$x_2 = a\,(t_2 - t_0) = .4078,$$
$$y_2 = a \operatorname{ch} t_2 = .4343.$$

From a series of four actual experiments the following coordinates for P_2 were obtained (see also part C, §5, below):

	b_0 (*cm.*)	y_2 (*cm.*)	x_2 (*cm.*)
	3.80	1.700	1.525
	3.80	1.625	1.525
	3.80	1.575	1.525
	3.80	1.625	1.525
average	3.80	1.630	1.525
Reduced to scale $b_0 = 1$	1	.4284	.4013
Theoretical value	1	.4360	.4078

8. Necessity for a stronger form of conditions (26) **and** (23). For the envelope, E, of the set $\mathfrak{E}_1$, the slope is $dy/dx = \operatorname{sh}\tau > 0$. Hence E has no singular point, and the stronger form of condition (26),

$$\tau_1 < t_0',$$

is necessary.* If $t_0 = t_0^*$, τ_2 must be equal to t_1'. Hence the stronger form, $t_0 > t_0^*$, of (23) is necessary.

* Bolza, loc. cit., §38, p. 204.

9. Summary of the necessary conditions. The minimizing system of curves must be composed of arcs of two catenaries $\mathfrak{E}_0$, $\mathfrak{E}_1$ having the x-axis as directrix, together with the normal $\mathfrak{E}_2$ to the x-axis from their point of intersection A_2.

$\mathfrak{E}_0$ and $\mathfrak{E}_1$ must be symmetrically situated with respect to $\mathfrak{E}_2$ in such a way that the three curves make angles of 120° with each other, as indicated by equations (8) and (12). The value of t_0 must be greater than t_0^*, where t_0^* is the negative value of t satisfying

$$\chi(t) = 2 + \log\sqrt{3},$$

or in other words, A_0 must lie between A_2 and its conjugate (in sense of §5 above) on the curve $\mathfrak{E}_0$. Furthermore A_1 must lie between A_2 and a point A_0' determined on $\mathfrak{E}_1$ by the equation

$$\chi(\tau_0') = \chi(t_0) - \log\sqrt{3}.$$

The point A_0' is said to be the conjugate point to A_0 on the broken extremal $(\mathfrak{E}_0, \mathfrak{E}_1)$.*

B. Sufficient Conditions.

We now suppose that for the extremal system $\mathfrak{E}_0\, \mathfrak{E}_1\, \mathfrak{E}_2$ the conditions described in the last section are all satisfied. Since the problem is regular and since $t_0 < t_2 < 0$ and $0 < \tau_2 < \tau_1$, it follows that Bolza's conditions I, II′, III′, IV′ are all satisfied for $\mathfrak{E}_0\, \mathfrak{E}_1\, \mathfrak{E}_2$ separately. Further, under the above conditions, a certain vicinity of A_2 $(a_2\, b_2)$,

$$|x_2 - a_2| < \delta, \quad |y_2 - b_2| < \delta,$$

can be assigned in which

$$\phi(a_2\, b_2) < \phi(x_2\, y_2),$$

where $\phi(x_2\, y_2)$ is the function defined in (5) above.

Lemma. The broken-extremal $\mathfrak{E}_0\, \mathfrak{E}_1$ is contained wholly within the region R_0 bounded by the ordinate through A_0 and the envelope F_0 of the set of extremals through A_0. The infinite segment of $\mathfrak{E}_0$ with initial point A_0 lies in R_0. Hence A_2 lies in R_0. Let A_3 be a point on $\mathfrak{E}_0$ symmetrical to A_2 with respect to the axis of $\mathfrak{E}_0$. Then the segment of $\mathfrak{E}_1$ with A_2 as initial point can be obtained by a lateral displacement of the arc $A_3\, \infty$ of $\mathfrak{E}_1$, bringing A_3 into coincidence with A_2, and $\mathfrak{E}_1$ is thus wholly within R_0.

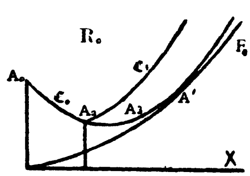

FIG. 4.

* The conditions denoted II′, III′, IV′ by Bolza, are all satisfied for $\mathfrak{E}_0$ and $\mathfrak{E}_1$ separately under the above conditions.

We now construct the envelope F_0 of the set of catenaries through A_0, and the envelope F_1 of the set through A_1, and let S be the region common to the regions R_0 and R_1 which they define. Then A_2 lies in S, and $\mathfrak{E}_0$ and $\mathfrak{E}_1$ lie in S.

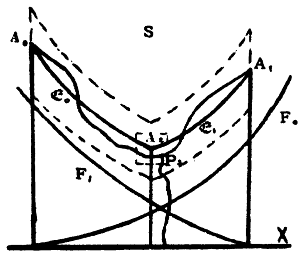

FIG. 5.

We can now take δ so small that the vicinity (δ) lies entirely in S and we can construct a neighborhood (ρ) of $\mathfrak{E}_0$ and $\mathfrak{E}_1$ also lying wholly in S. Let $P_2(x_2, y_2)$, be any point common to (δ) and (ρ) and let $\bar{\mathfrak{E}}_0$, $\bar{\mathfrak{E}}_1$ be catenaries joining P_2 with A_0 and A_1 respectively, and $\mathfrak{C}_0$ and $\mathfrak{C}_1$ any other ordinary curves in (ρ) joining P_2 with A_0 and A_1 respectively. Let $\bar{\mathfrak{E}}_2$ be the normal from P_2 to the x-axis, and $\mathfrak{C}_2$ any ordinary curve from P_2 to the x-axis. Then

$$I(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) = \phi(a_2, b_2),$$

$$I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2) = \phi(x_2, y_2),$$

and hence

$$I(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) < I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2).$$

But $\quad I(\bar{\mathfrak{E}}_0) < I(\mathfrak{C}_0), \qquad I(\bar{\mathfrak{E}}_1) < I(\mathfrak{C}_1), \qquad I(\bar{\mathfrak{E}}_2) < I(\mathfrak{C}_2),$

and therefore

$$I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2) < I(\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2).$$

Hence we have

$$I(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) < I(\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2).$$

*The conditions given in A, §9 are therefore not only necessary but sufficient conditions that the extremal-system $\mathfrak{E}_0$ $\mathfrak{E}_1$ $\mathfrak{E}_2$ furnish a relative minimum with respect to the admissible curves defined in A, §1, above.**

* Our problem is similar to those considered by Carathéodory in his Göttingen Dissertation, "*Ueber die diskontinuirlichen Lösungen in der Variationsrechnung.*" The fact that we have three curves meeting in a point makes a separate discussion necessary.

C. Uniqueness of Extremal-Systems Within a Field.

We shall study now more carefully the region in which a discontinuous solution exists.

1. The corner curve, K. We first consider the curve K, which is the locus of the point P_2. From (29),

$$(31) \qquad K: \quad \begin{aligned} y_2 &= \frac{2}{\sqrt{3}}\, a, \\ x_2 &= a_0 + a\left(-\log\sqrt{3} + \mathrm{ch}^{-1}\frac{b_0}{a}\right), \end{aligned}$$

where $\mathrm{ch}^{-1}(b_0/a)$ is positive, and $t_0 = -\,\mathrm{ch}^{-1}(b_0/a)$. We may write the single equation for K,

$$(32) \qquad K: \qquad x_2 - a_0 = \frac{\sqrt{3}}{2}\, y_2\left[\mathrm{ch}^{-1}\frac{2b_0}{\sqrt{3}\,y_2} - \log\sqrt{3}\right],$$

where the positive sign of the function $\mathrm{ch}^{-1}(2b_0/\sqrt{3}\,y_2)$ is to be taken. Here x_2 appears as a single-valued continuous function of y_2 defined for the range $0 < y_2 \overline{\gtrless}\, b_0$, and x_2 approaches the value a_0 as y_2 approaches zero. Further, if $y_2 = b_0$, we have $x_2 = a_0$.

We compute from (31) the derivatives

$$(33) \qquad \begin{aligned} \frac{dx_2}{dy_2} &= \frac{\dfrac{dx_2}{da}}{\dfrac{dy_2}{da}} = \frac{\sqrt{3}}{2}\left\{\mathrm{ch}^{-1}\frac{b_0}{a} - \log\sqrt{3} - \frac{b_0}{\sqrt{b_0^2 - a^2}}\right\}, \\ \frac{d^2x}{dy_2dy_2} &= \frac{\dfrac{dy_2}{da}\cdot\dfrac{d^2x_2}{da^2} - \dfrac{dx_2}{da}\cdot\dfrac{d^2y_2}{da^2}}{\left(\dfrac{dy}{da}\right)^3} = -\,\frac{3b_0^3}{4a(b_0^2 - a^2)^{3/2}} < 0. \end{aligned}$$

Hence $\dfrac{dx_2^{\prime}}{dy_2}$ is a decreasing function and the curve K is concave to the line $x = a_0$.

Furthermore $\dfrac{dx_2}{dy_2}$ approaches $+\infty$ as y_2 approaches 0, takes the value -2 for $y_2 = b_0$, and the value zero when $\chi(t_0) = \log\sqrt{3}$, where $b_0 = a\,\mathrm{ch}\,t_0$.

This value of t_0 is greater than t_0^* where $\chi(t_0^*) = 2 + \log \sqrt{3}$, since $\chi(t)$ is a decreasing function, continuous except at $t = 0$, and t_0 and t_0^* are negative. We find approximately $t_0 = -1.63$ and $t_0^* = -3.55$.

From (31) the following table, used for plotting the curve K in figure 6, is obtained:

t_0	a	x_2	y_2
-3.55	.060	.180	.070
-3.00	.098	.240	.113
-2.46	.162	.316	.187
-2.00	.262	.380	.302
-1.50	.416	.394	.480
-1.20	.556	.361	.642
-1.00	.652	.293	.753
-0.75	.774	.155	.894
-0.55	.866	0.000	1.000

As t_0 decreases from $-\log\sqrt{3}$ to $-t_0^*$, A_2 describes the curve K from A_0 to the fixed point L for which $x_2 = .180$, $y_2 = .070$.

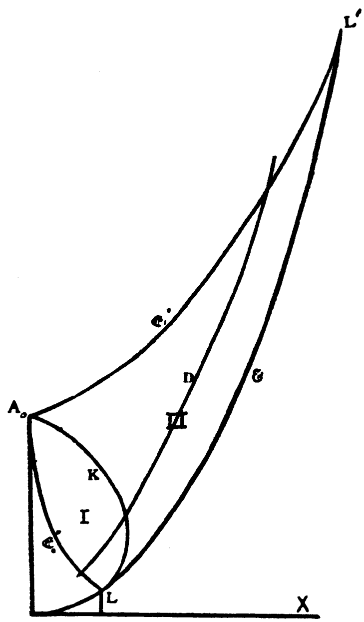

FIG. 6.

2. The envelope E. We now consider the envelope E of the catenaries $\mathfrak{E}_1(a)$, the coordinates of which satisfy the conditions

$$(34) \qquad E: \quad \begin{aligned} x &= a(\tau - t_0 - \log 3) + x_0, \\ y &= a \operatorname{ch} \tau, \end{aligned}$$

where

$$b_0 = a \operatorname{ch} t_0, \qquad \chi(\tau) + \log 3 - \chi(t_0) = 0.$$

Since $\chi(t)$ is a single-valued decreasing function, continuous except at $t = 0$, the equation $\chi(t) = c$ has for any assigned c one and only one negative solution and one and only one positive solution. Hence, in the realm

$$t_0^* < t \gtreqless - \log \sqrt{3}, \qquad \log \sqrt{3} \gtreqless \tau < t_0',$$

the values x, y, t_0, and τ above are single-valued continuous functions of a. We find

$$\frac{dx}{da} = \coth \tau - \coth t_0 + a\left(\frac{\partial \tau}{\partial a} - \frac{\partial t_0}{\partial a}\right),$$

$$\frac{dy}{da} = \operatorname{ch} \tau + a \operatorname{sh} \tau \frac{\partial \tau}{\partial a}.$$

But

$$\frac{\partial t_0}{da} = -\frac{1}{a} \coth t_0,$$

$$\frac{\partial t}{\partial a} = \frac{\coth^2 t_0}{\coth^2 \tau} \frac{\partial t_0}{\partial a} = -\frac{\coth^3 t_0}{a \coth^2 \tau}.$$

Hence

$$\frac{dx}{da} = \frac{\coth^3 \tau - \coth^3 t_0}{\coth^2 \tau} > 0,$$

$$\frac{dy}{da} = \frac{\operatorname{sh}^3 \tau}{\operatorname{ch}^2 \tau} (\coth^3 \tau - \coth^3 t_0) > 0,$$

and, as was found above,

$$\frac{dy}{dx} = \operatorname{sh} \tau.$$

The slope of E is therefore everywhere positive.
Furthermore

$$\frac{d^2 y}{dx^2} = \operatorname{ch} \tau \frac{d\tau}{dx}.$$

Since

$$\tau = \frac{x - x_0}{a} + \log 3 + t_0,$$

we obtain

$$\frac{d\tau}{dx} = \frac{1}{a} - \frac{x - x_0}{a^2}\frac{da}{dx} + \frac{dt_0}{dx}$$

$$= \frac{1}{a}\left[1 - \frac{da}{dx}\,(\coth t_0 + \coth\tau - \coth t_0)\right]$$

$$= \frac{1}{a}\left[1 - \frac{\coth^3\tau}{\coth^3\tau - \coth^3 t_0}\right]$$

$$= -\frac{1}{a}\,\frac{\coth^3 t_0}{\coth^3\tau - \coth^3 t_0}\,.$$

Therefore

$$\frac{d^2y}{dx_2} = -\frac{\operatorname{ch}\tau}{a}\,\frac{\coth^3 t_0}{\coth^3\tau - \coth^3 t_0} > 0.$$

Hence E is everywhere convex to the x-axis. For the initial values, $a_0 = 0$, $b_0 = 1$, we have $a = 1/\operatorname{ch} t_0$. Using $\chi(t_0)$, $\chi(\tau)$, we compute the following table for points of E:

t_0	$\chi(t_0)$	$\chi(\tau)$	(τ)	a	x	y
-3.55	2.55	1.45	0.55	.060	0.18	0.07
-3.00	2.00	0.90	0.71	.098	0.24	0.11
-2.50	1.49	0.39	0.95	.162	0.42	0.24
-2.00	0.96	-0.14	1.29	.262	0.56	0.50
-1.50	0.40	-0.65	1.73	.416	0.89	1.20
-1.20	0.00	-1.10	2.15	.556	1.25	2.53
-1.00	-0.31	-1.41	2.41	.652	1.53	3.68
-0.75	-0.82	-1.92	2.92	.774	2.00	7.70
-0.55	-1.45	-2.55	3.55	.866	2.60	15.05

The curve E appears in figure 6. It touches the corner curve K in the point $L(x = .180,\ y = .07)$, and the particular curve $\mathfrak{E}_1^*$, for which $t_0 = -\log\sqrt{3}$ in the point L' $(x = 2.60,\ y = 15.05)$ where $\tau = \tau_1^*$.

3. The field of broken-extremals $[\mathfrak{E}_0(a), \mathfrak{E}_1(a)]$ for which $t_0^* < t_0 \overline{\gtrless} -\log\sqrt{3}$. We have now a region I (in figure 6) bounded by the curves $\mathfrak{E}_0^*(t_0 = t_0^*)$ and K, and a second region II (in figure 6) bounded by K, $\mathfrak{E}_1^*$, and E. Every line parallel to the x-axis cuts the boundary of I twice (or not at all), once on $\mathfrak{E}_0^*$ and once on K, and cuts the boundary of II twice (or not at all), once on K or $\mathfrak{E}_1^*$ and once on E, for each of these curves is expressible in the form $x = \phi(y)$, where ϕ is a single-valued function.

The set of extremals $\mathfrak{E}_0(a)$, $(a_0^* < a < a_2)$, of the region I, where $a_0^* = \dfrac{b_0}{\text{ch}\, t_0^*}$ and $a_2 = \frac{1}{2}\sqrt{3}\, b_0$, are expressible in the form

$$x = x(y, a),$$

where x, x_y, x_a, x_{yy}, x_{ya} are single-valued continuous functions in the region I. Further

$$x_a = \chi(t_0) - \chi(t) > 0 \text{ in I (unless } t = t_0).$$

Consider any point $[x(y, a), y]$ in I. If we give to y any fixed value y_3 of I and let a increase continuously from a^* of $\mathfrak{E}_0^*$ to $a_k = \frac{1}{2}\sqrt{3}\, y_3$ at K, $x(y_3, a)$ increases continuously from $x(y_3, a^*)$ to $x(y_3, a_k)$, and therefore passes once and but once through every intermediate value. Hence, if a_3 be any value of a in (a^*, a_k) and we put $x(y_3, a_3) = x_3$, then the equation $x(y_3, a) = x_3$ has in (a^*, a_k) no other solution but $a = a_3$. Through every point (x_3, y_3) of I there passes, therefore one and but one extremal of the set. The region I, is thus a field (improper) of extremals $\mathfrak{E}_0(a)$.*

The set of extremals $\mathfrak{E}_1(a)$, $(a_0^* < a < a_2)$, (29), of II is also expressible in the form

$$x = x(y, a),$$

where x, xy, x_a, x_{yy}, x_{ya} are single-valued continuous functions in the region II. Further [see (20)],

$$x_a = \chi(t_0) - \chi(\tau) - \log 3 < 0, \text{ in II}.$$

By an argument similar to the one above we may show that the region II is simply covered by the extremals $\mathfrak{E}_1$.

Under these circumstances, we say that *the region R, composed of the*

---

* Bolza, loc. cit., §19 *a, b*.

regions I *and* II, *constitutes a field simply covered by the broken-extremals* $\mathfrak{E}_0(a)$, $\mathfrak{E}_1(a)$.*

4. Comparison of the system $(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$ **with the Goldschmidt solution.**† For the value of the integral over the system $(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$ we find

$$J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) = 2\pi a^2 \int_{t_0}^{-\log\sqrt{3}} \operatorname{ch} t\, dt + 2\pi a^2 \int_{\log\sqrt{3}}^{\tau_1} \operatorname{ch}^2\tau\, dt + \tfrac{4}{3}\pi a^2,$$

$$= \frac{\pi}{2} a^2 \left\{ (\operatorname{sh} 2t + 2t) \Big|_{t_0}^{\tau_1} - 4 \log\sqrt{3} \right\}.$$

For the Goldschmidt solution, consisting of the two ordinates through A_0 and A_1 together with the x-axis, the corresponding value is

$$J_G = \pi(b_0^2 + b_1^2) = 4a^2 (\operatorname{ch} t_0^2 + \operatorname{ch}^2 \tau_1),$$

and consequently

$$(35) \quad J_G - J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$$

$$= \frac{\pi}{2} a^2 \{ 2\operatorname{ch}^2 t_0 + \operatorname{sh} 2t_0 + 2t_0 + 2\operatorname{ch}^2\tau_1 - \operatorname{sh} 2\tau_1 - 2\tau_1 + 2\log 3 \}$$

$$= \frac{\pi}{2} a^2 \{ 2t_0 + 1 + e^{2t_0} + 2\log 3 - (2\tau_1 - 1 - e^{-2\tau_1}) \}.$$

Let now A_0 be fixed, and A_1 be movable, on a given extremal-system. Then t_0 is fixed, and τ_1 variable, and we consider the function

$$\phi(\tau_1) = 2\tau_1 - 1 - e^{-2\tau_1}.$$

This function has a positive derivative,

$$\phi'(\tau_1) = 2 + 2e^{-2\tau_1} > 0.$$

Hence, as τ_1 increases from its smallest value $\log\sqrt{3}$ to ∞, the difference (35) decreases continuously. On the arc A_0L' of the curve $\mathfrak{E}_1^*$ we find that if A_1 is coincident with A_0, τ_1 will be equal to $\log\sqrt{3}$ and

$$J_G - J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) = \tfrac{2}{3}\pi a^2 > 0.$$

---

* Compare Carathéodory, loc. cit., §11, where a similar field (fig. 14) appears for a discontinuous solution of the usual kind.

† Goldschmidt, *Prize Essay*, 1830.

If A_1 is coincident with L', τ_1 will be equal to $\tau_1^{\cdot} = 3.55$ and we have

$$J_G - J(\mathfrak{E}_2, \mathfrak{E}_1, \mathfrak{E}_2) < 0,$$

the difference being approximately equal to $-\frac{1}{2}(3.67)(\pi a^2)$. Hence the expression (35) vanishes at some point of $\mathfrak{E}_1^{\cdot}$ between A_0 and L'. The totality of points for which this is true on the extremals $\mathfrak{E}_1$ of the field form a continuous curve D determined by the equations

$$D: \quad \begin{aligned} x &= a\tau_1 - a \log 3 + a_0 - at_0, \\ y &= a \operatorname{ch} \tau_1, \end{aligned}$$

where

$$b_0 = a \operatorname{ch} t_0,$$

and

$$2t_0 + 1 + e^{2t_0} + 2 \log 3 = (2\tau_1 - 1 - e^{-2\tau_1}) = z.$$

For the values, $a_0 = 0$, $b_0 = 1$, we construct the following table for the curve D:

t_0	z	τ_1	$\tau_1 - t_0 - \log 3$	a	x	y
− 3.0	− 2.80	− .16	1.10	.098	.108	.10
− 2.5	− 1.79	.08	1.32	.162	.216	.16
− 2.0	− .78	.35	1.25	.262	.327	.28
− 1.5	.25	.74	1.14	.416	.476	.54
− 1.2	.89	1.01	1.11	.556	.618	.87
− 1.0	1.34	1.20	1.11	.652	.724	1.18
− .75	1.93	1.49	1.14	.774	.884	1.79
− .50	2.43	1.73	1.18	.866	1.022	2.52

The curve D (see figure 6) divides R into two regions, one adjacent to A_0, in which

$$J_G - J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$$

is positive, and the other adjacent to L', in which this difference is negative. On D itself, J_G and $J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$ are equal.

5. Tallqvist's results. Experimental determination of the limit of stability. In the first of the two papers referred to on page 56, Tallqvist starts from the problem to construct a minimal surface passing

through a given circle and meeting a given plane parallel to the circle at a given angle α. He derives the solution from certain formulae given by H. A. Schwarz.* He then infers from physical considerations that the angle α must be 60°, and finally gives the following rule for the determination of the limit of stability:

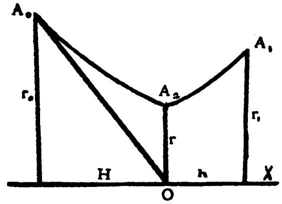

FIG. 7.

Let r_0 and r_1 $(r_0 > r_1)$ be the radii of the rings. Let $A_0 A_2$ and $A_2 A_1$, be two catenaries which meet a normal OA_2 to their common directrix, at an angle of 60°. Let H be the projection on the directrix of the film $A_0 A_2$, and h that of $A_2 A_1$. Then H is determined by the condition that the tangent to $A_0 A_2$ at $A_0 (y_0 = r_0)$ must pass through the foot of the normal OA_2. This determines the catenaries (see p. 15 above), and h is determined by the point A_1 of the catenary $A_2 A_1$ whose ordinate is r_1.

This leads to the following formulas:

$$\log \frac{2r_0 + \sqrt{4r_0^2 - 3r^2}}{3r} = \frac{2r_0}{\sqrt{4r_0^2 - 3r^2}} = \frac{2H}{r\sqrt{3}},$$

$$h = r\,\frac{\sqrt{3}}{2}\, ln\, \frac{2r_1 + \sqrt{4r_1^2 - 3r^2}}{3r},$$

from which it is found that

$$\frac{r}{r_0} = 0.436047,$$

$$\frac{H}{r_0} = 0.407824.$$

The quotient h/r_0 can be found for any values r_0, r_1 $(r_1 \leqq r_0)$.

In the special case where the two rings are equal, Tallqvist's rule agrees exactly with our own result given in A, §7. In the general case, however, for the limit of stability when the two rings are unequal, Tallqvist's rule gives

* H. A. Schwarz, *Miscellen aus dem gebiete der Minimalflächen.*

a different value from the results obtained in A, §9.* Tallqvist's experiments for this case are not decisive as the two rings which he used were too nearly equal in size. We give below his results for unequal rings compared with his formulas and the results found from the formulae of the present paper, where b_0 and y are the radii of the rings:

	b_0	y	$x - a_0 = H + h$
Tallqvist's	22.455 *mm*	22.402 *mm*	18.432
experimental values (21 trials)	1.	.99766	.82086
Tallqvist's	22.455	22.402	18.296
theoretical values	1.	.99766	.81480
Theoretical values from tables given above	1.	.99766	.81480

Since these results are inconclusive it seemed desirable to carry out a new series of experiments in which the two rings should be of quite different diameter. The experiments were conducted as follows (see figure 8):

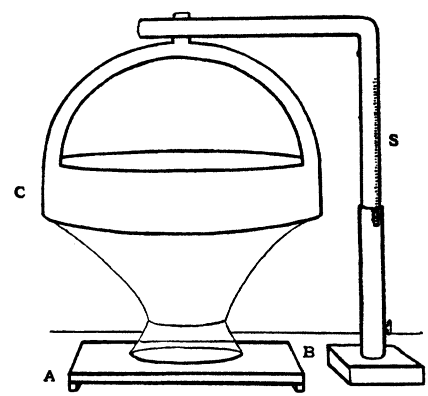

FIG. 8.

A square plate of brass AB with a circular opening, the latter beveled to a knife-edge at the upper surface, was put in horizontal position by means of a spirit level and held rigidly in place. Above this was suspended a ring C

* Compare also C §2, formulas (34).

made from a band of heavy brass, beveled at the bottom on the inside to a knife-edge. This ring was attached to the arm of a stand S, which was provided with a screw to give smooth vertical movement and a vernier scale to ensure accurate measurement.

The upper face of the brass plate was covered with a film, the ring lowered to it and withdrawn. At first there appeared beside the horizontal film across the circular opening of the plate a single catenoid extending from the ring to the plate. When the ring was withdrawn so far that the lower margin of the catenoid became as small as the circular opening, the films were transformed into the desired system. The height was then adjusted until the limit of stability was found. The diameters of the two circles were measured with calipers. The following are the results, where D_0 and D_1 are the diameters of the two rings, H the distance between them, or the height of the film:

	D_0	D_1	H_1
	87.7 *mm*	51.15 *mm*	27.3 *mm*
	87.8	51.15	27.25
	87.4	51.10	27.25
	87.0	51.20	27.3
	87.1	51.10	27.4
	87.6	51.15	27.35
	87.8	51.15	27.4
	87.7	51.15	27.35
	87.0	51.1	27.35
	87.5	51.15	27.3
			27.5
			27.3
			27.35
			27.3
			27.3
			27.3
Average	87.52	51.14	27.33

Then, if R_0 and R_1 represent the radii of the two circles, and if we reduce to the scale for which $R_0 = 1$, we have $R_0 = 1$, $R_1 = .5844$, and $H_1 = .6244$.

This value of H_1 can readily be compared with that given by formulas C', §2 (34). We have

$$a_0 = 0, \qquad b_0 = 1 = a \operatorname{ch} t_0,$$

$$\chi(t_0') - \chi(t_0) + \log 3 = 0.$$

Then

$$y = .5844 = a \operatorname{ch}(t_0'),$$

$$x = H_1 = a(t_0' - t_0 - \log 3).$$

These equations have, for t_0 negative and t_0' positive, a unique solution, found by the method of approximation :

$$a = .2835, \quad t_0 = 1.933, \quad t_0' = 1.3515, \quad x = H_1 = .6195.$$

Tallqvist's formulas give $x - a_0 = H + h = .5795$. By the formulas here given, the error is 0.8 %, by Tallqvist's 7.2 %. From these results, it may be inferred that Tallqvist's theoretical determination of the limit of stability is wrong, and that the minimum problem which nature solves in the experiment of the two rings is actually the one which we have formulated in the introduction.

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